Definition (Dual Space)

Let $(X, \|\cdot\|)$ be a normed space. Its dual space $X^* := B(X, \mathbb{K})$ is the space of all the bounded linear maps from X to $\mathbb{K}$ with the operator norm $\|T\| := \sup\{|Tx| : \|x\| \leq 1\}$.

Example

For $1 \leq p < \infty$, $\ell^p := \{(x(k)) : x(k) \in \mathbb{K}, \sum_{n=1}^{\infty} |x(k)|^p < \infty\}$.

$\ell^\infty := \{(x(k)) : x(k) \in \mathbb{K}, \sup_i |x(k)| < \infty\}$.

(a) $(\ell^1)^* = \ell^\infty$

Proof: Define $T : \ell^\infty \to (\ell^1)^*$ by $(Tx)(y) = \sum_{i=1}^{\infty} x(k) y(k)$ for any $x \in \ell^\infty$ and $y \in \ell^1$.

(i) T is well defined.

For any $x \in \ell^\infty$, $y \in \ell^1$,

$$|Tx(y)| = \left|\sum_{i=1}^{\infty} x(k) y(k)\right| \leq \sum_{i=1}^{\infty} |x(k)||y(k)|$$

$$\leq \sup_i |x(k)| \sum_{i=1}^{\infty} |y(k)|$$

$$= \|x\|_\infty \|y\|_1$$

(ii) T is an isometry.

By (i), $\|Tx\|_{op} \leq \|x\|_\infty$ for any $x \in \ell^\infty$.

We wish $\|Tx\|_{op} \geq \|x\|_\infty$ for any $x \in \ell^\infty$.

For any $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$|x(k_0)| > \|x\|_\infty - \varepsilon$

Take $y \in \ell^1$ with $y(k_0) = 1$ and $y(k) = 0$ for $k \neq k_0$

Then $\|y\|_1 = \sum_{k=1}^{\infty} |y(k)| = |y(k_0)| = 1$.

And
$$\|Tx\|_{op} \geq |Tx(y)| = \left|\sum_{k=1}^{\infty} x(k) y(k)\right|$$
$$= |x(k_0) y(k_0)|$$
$$= |x(k_0)|$$
$$> \|x\|_\infty - \varepsilon$$

Letting $\varepsilon \to 0$ gives $\|Tx\|_{op} \geq \|x\|_\infty$.

(iii) T is surjective.

Pick any $\phi \in (\ell^1)^*$. We want to find an $x \in \ell^\infty$ such that $Tx = \phi$.

Put $x(k) = \phi(e_k)$. By definition of bounded linear operator, $x \in \ell^\infty$.

For any $y \in \ell^1$.

$$\begin{aligned} Tx(y) &= \sum_{k=1}^{\infty} x(k)\, y(k) \\ &= \sum_{k=1}^{\infty} \phi(e_k)\, y(k) \\ &= \phi\Big(\sum_{k=1}^{\infty} y(k)\, e_k\Big) && \text{linearity} \\ &= \phi(y) && \text{Schauder base} \end{aligned}$$

Hence, $Tx = \phi$.

$\square$

(b) For $1 < p < \infty$, $(\ell^p)^* = \ell^q$ where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof: Define $T : \ell^q \to (\ell^p)^*$ by $Tx(y) = \sum_{k=1}^{\infty} x(k)\, y(k)$ for any $x \in \ell^q$ and $y \in \ell^p$.

(i) T is well defined.

For any $x \in \ell^q$ and $y \in \ell^p$

$$
\begin{aligned}
|Tx(y)| &= \left|\sum_{k=1}^{\infty} x(k)\, y(k)\right| \\
&\le \sum_{k=1}^{\infty} |x(k)|\,|y(k)| \\
&\le \|x\|_q \|y\|_p \qquad \text{Hölder's inequality} \\
&< \infty
\end{aligned}
$$

(ii) T is an isometry.

By (i), $\|Tx\|_{op} \le \|x\|_q$ for any $x \in \ell^q$

We wish $\|Tx\|_{op} \ge \|x\|_q$ for any $x \in \ell^q$

For any $x \in l_q$, write $x(k) = |x(k)| e^{i\theta_k}$.

Put $y(k) = |x(k)|^{q-1} e^{-i\theta_k} / \|x\|_q^{q-1}$

Then
$$
\begin{aligned}
\sum_{k=1}^{\infty} |y(k)|^p &= \|x\|_q^{-(q-1)p} \sum_{k=1}^{\infty} |x(k)|^{(q-1)p} \\
&= \|x\|_q^{-q} \sum_{k=1}^{\infty} |x(k)|^q \qquad p = \frac{q}{q-1} \\
&= 1
\end{aligned}
$$

Therefore, $y \in \ell^p$ and $\|y\|_p = 1$.

And $\|Tx\|_{op} \geq |Tx(y)|$

$$= \left|\sum_{k=1}^{\infty} x(k)\, y(k)\right|$$

$$= \|x\|_q^{-(q-1)} \sum_{k=1}^{\infty} |x(k)|\, e^{i\theta_k} |x(k)|^{q-1} e^{-i\theta_k}$$

$$= \|x\|_q^{-(q-1)} \sum_{k=1}^{\infty} |x(k)|^q$$

$$= \|x\|_q^{-(q-1)} \|x\|_q^q$$

$$= \|x\|_q$$

(iii) T is surjective.

Pick any $\phi \in (\ell^p)^*$. We want to find an $x \in \ell^q$ such that $Tx = \phi$.

Put $x(k) = \phi(e_k)$.

Consider $y_n(k) = \begin{cases} |\phi(e_k)|^q / \phi(e_k) & , k \leq n. \\ 0 & , k \geq n. \end{cases}$

Then $y_n \in \ell^p$.

$$\sum_{k=1}^{n}|x(k)|^q = \sum_{k=1}^{n}|\phi(e_k)|^q$$

$$= |\phi(y_n)|$$

$$\leq \|\phi\|_{op} \|y_n\|_p$$

$$= \|\phi\|_{op}\left(\sum_{k=1}^{n}|\phi(e_k)|^{(q-1)p}\right)^{\frac{1}{p}}$$

$$= \|\phi\|_{op}\left(\sum_{k=1}^{n}|\phi(e_k)|^{q}\right)^{\frac{1}{p}}$$

$$\Longrightarrow \left(\sum_{k=1}^{n}|\phi(e_k)|^q\right)^{\frac{1}{q}} \leq \|\phi\|_{op} \qquad \frac{1}{q} = 1 - \frac{1}{p}$$

Letting $n \to \infty$ gives $\left(\sum_{k=1}^{\infty}|x(k)|^q\right)^{\frac{1}{q}} = \left(\sum_{k=1}^{\infty}|\phi(e_k)|^q\right)^{\frac{1}{q}} \leq \|\phi\|_{op}$

Therefore, $x \in \ell^q$.

And $$Tx(y) = \sum_{k=1}^{\infty} x(k)y(k)$$

$$= \sum_{k=1}^{\infty}\phi(e_k)y(k)$$

$$= \phi\left(\sum_{k=1}^{\infty} y(k)e_k\right) \qquad \text{Schauder base}$$

$$= \phi(y) \qquad \text{for any } y$$

Hence, $Tx = \phi$.

$\square$